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FURTHER MATHEMATICS

9231/21

Paper 2 Further Pure Mathematics 2

October/November 2024

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer all questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do not write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [].

This document has 20 pages. Any blank pages are indicated.

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Find the set of values of k for which the system of equations

$$x+5y+6z = 1,$$

 $kx+2y+2z = 2,$
 $-3x+4y+8z = 3,$

| has a unique solution and interpret this situation geometrically. | [4] |
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2 It is given that

(a)

| $x = 1 + \frac{1}{t}$ | and | $y = \cos^{-1} t$ | for $0 < t < 1$. |
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| Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2}{\sqrt{1-t^2}}$. | [2] |
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| Show that $\frac{d^2y}{dx^2} = -t^a(1-t^2)^b(2-t^2)$, where a and b are constants to be determined. | [4] |
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[6]



- A curve has equation $y = e^x$ for $\ln \frac{4}{3} \le x \le \ln \frac{12}{5}$. The area of the surface generated when the curve is rotated through 2π radians about the x-axis is denoted by A.
 - (a) Use the substitution $u = e^x$ to show that

| $A = 2\pi \int_{\frac{4}{3}}^{\frac{12}{5}} \sqrt{1 + u^2} \mathrm{d}u.$ | [2] |
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(b) Use the substitution $u = \sinh v$ to show that

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 $A = \pi \left(\frac{904}{225} + \ln \frac{5}{3} \right).$

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4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}.$$

| (a) | Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A and state the corresponding eigenvalue. | [2] |
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| (b) | Show that the characteristic equation of A is $\lambda^3 - 19\lambda - 30 = 0$ and hence find the ot eigenvalues of A . | ther [3] |
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| (c) | Use the c | haracterist | ic equation | on of |

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| Use the characteristic equation of A to find A^{-1} . | [4] |
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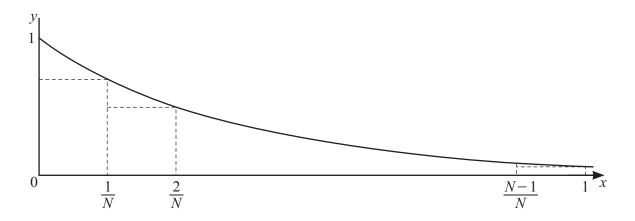
5 Find the particular solution of the differential equation

$$6\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + x = t^2 + t + 1,$$

| given that, when $t = 0$, $x = 12$ and $\frac{dx}{dt} = -6$. | [10] |
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The diagram shows the curve with equation $y = \left(\frac{1}{2}\right)^x$ for $0 \le x \le 1$, together with a set of N rectangles each of width $\frac{1}{N}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{2}\right)^x dx > L_N$, where

| $L_N = \frac{1}{2N(2^{\frac{1}{N}} - 1)}.$ | [4] |
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| | Use a similar method to find, in terms of N , an upper bound U_N for $\int_0^1 \left(\frac{1}{2}\right)^x dx$. |
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| F | Find the least value of N such that $U_N - L_N \le 10^{-3}$. |
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| (d) | Given that $\int_0^1 \left(\frac{1}{2}\right)^x dx = \frac{1}{2 \ln 2}$, use the value of N found in part (c) to find upper and lower bounds for $\ln 2$. |
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(a) Show that an appropriate integrating factor for 7

$$\sqrt{x^2 + 16} \, \frac{\mathrm{d}y}{\mathrm{d}x} + y = x\sqrt{x^2 + 16}$$

| is $\frac{1}{4}x + \frac{1}{4}\sqrt{x^2 + 16}$. | [4] |
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(b) Hence find the solution of the differential equation

| $\sqrt{x^2 + 16}$ | $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x$ | $\sqrt{x^2+16}$ |
|-------------------|---|-----------------|
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| for which $y = 6$ when $x = 3$. | [6] |
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8 (a) By considering the binomial expansion of $\left(z + \frac{1}{z}\right)^7$, where $z = \cos \theta + i \sin \theta$, use de Moivre's theorem to show that

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 $\cos^7 \theta = a \cos 7\theta + b \cos 5\theta + c \cos 3\theta + d \cos \theta,$ where a, b, c and d are constants to be determined.

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[5]



Let
$$I_n = \int_0^{\frac{1}{4}\pi} \cos^n \theta \, d\theta$$
.

(b) Show that

| $nI_n = 2^{-\frac{1}{2}n} + (n-1)I_{n-2}.$ | [4] |
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| (a) Haing the regults given in nexts (a |

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